Analytic Continuation

of

Chern-Simons Theory

Analytic Continuation Of Chern-Simons Theory

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Abstract

The title of this article refers to analytic continuation of three-dimensional Chern-Simons gauge theory away from integer values of the usual coupling parameter k, to explore questions such as the volume conjecture, or analytic continuation of three-dimensional quantum gravity (to the extent that it can be described by gauge theory) from Lorentzian to Euclidean signature. Such analytic continuation can be carried out by generalizing the usual integration cycle of the Feynman path integral. Morse theory or Picard-Lefschetz theory gives a natural framework for describing the appropriate integration cycles. An important part of the analysis involves flow equations that turn out to have a surprising four-dimensional symmetry. After developing a general framework, we describe some specific examples (involving the trefoil and figure-eight knots in S^3). We also find that the space of possible integration cycles for Chern-Simons theory can be interpreted as the "physical Hilbert space" of a twisted version of $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions.

Holomorphic Floer theory I: exponential integrals in finite and infinite dimensions

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Abstract

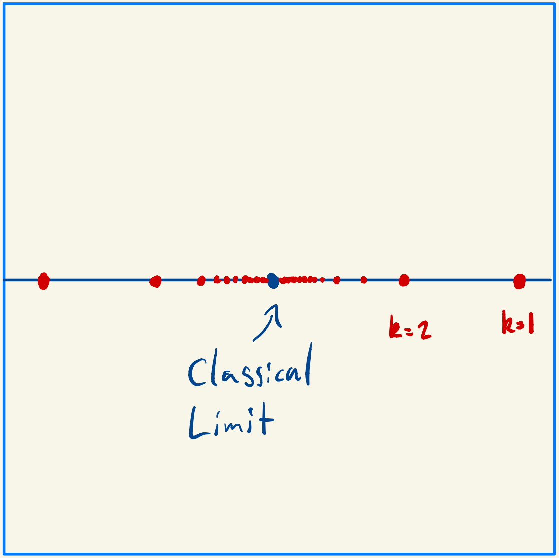
In the first of the series of papers devoted to our project "Holomorphic Floer Theory" we discuss exponential integrals and related wall-crossing structures. We emphasize two points of view on the subject: the one based on the ideas of deformation quantization and the one based on the ideas of Floer theory. Their equivalence is a corollary of our generalized Riemann-Hilbert correspondence. In the case of exponential integrals this amounts to several comparison isomorphisms between local and global versions of de Rham and Betti cohomology. We develop the corresponding theories in particular generalizing Morse-Novikov theory to the holomorphic case. We prove that arising wall-crossing structures are analytic. As a corollary, perturbative expansions of exponential integrals are resurgent. Based on a careful study of finite-dimensional exponential integrals we propose a conjectural approach to infinite-dimensional exponential integrals. We illustrate this approach in the case of Feynman path integral with holomorphic Lagrangian boundary conditions as well as in the case of the

We have $Z_{G,k}(M^3)$ [WRT]

G compact lie group(, simple)

$k \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$

<u>Def</u> $\hbar = \frac{1}{k}$



Question:

- Path integral def for Jones poly? (Not just evaluations at $q = e^{\frac{2\pi i}{k}}$)

- Is there an exact expression $Z_{G,k}(M) = \sum n_i Z^{(i)}_{G_{\mathbb{C}},k}(M)$?

  components of $G_{\mathbb{C}}$ character variety of M

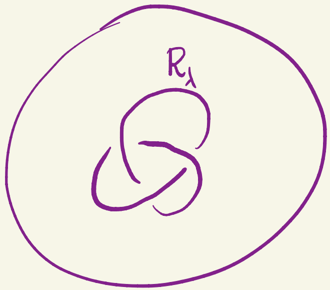
Observation:

- Asymptotics of $Z_{SU(2),k}(S^3, (K, R_\lambda))$

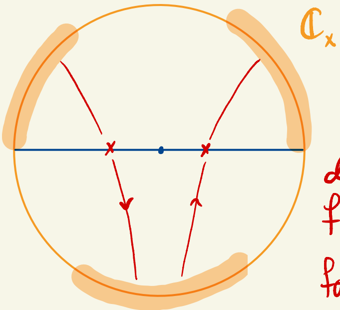
  as $k \to \infty$, $\lambda \to \infty$

  involve hyperbolic volume. $\mathrm{Vol}(M) = |CS(\text{Geometric rep } \pi_1(M) \to SL_2\mathbb{C})|$

  $M = S^3 \setminus K$



$$Z(\hbar) = \int_{\mathbb{R}} e^{-S/\hbar}\, dx \qquad S = i\left(\frac{x^3}{3} - x\right)$$



descending flow lines for $\mathrm{Re}(-S/\hbar)$

$$B := (S/\hbar)^{-1}\left([\text{"}1000\text{"}, \infty) \right)$$

$$[\mathbb{R}] \in H_1(\mathbb{C}_x, B)$$

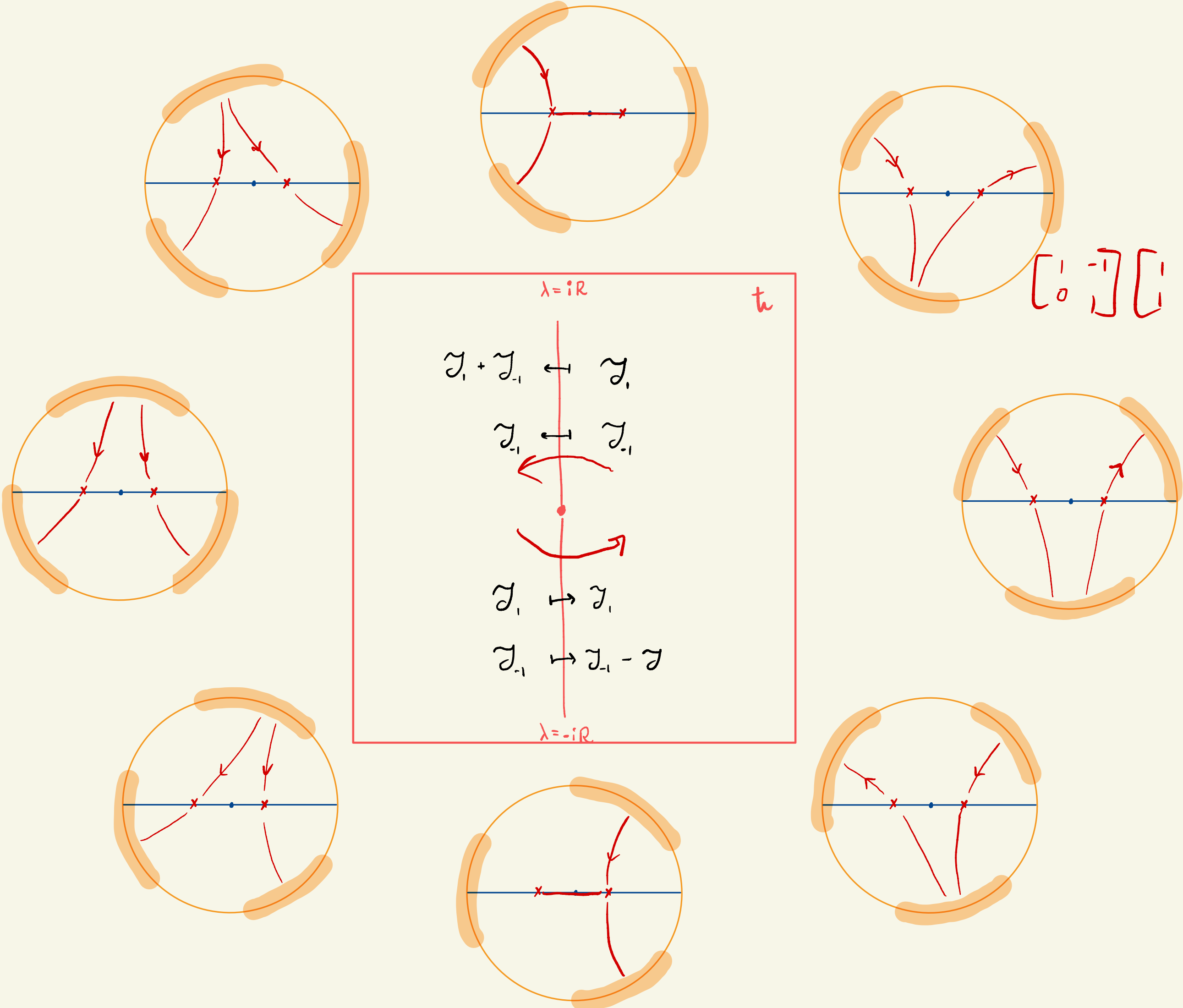
$$\mathrm{Crit}\left(\frac{x^3}{3} - x\right) = \{1, -1\}$$

$$[\mathbb{R}] = \mathcal{J}_1 + \mathcal{J}_{-1}$$

lefschetz thimbles

$$Z(\hbar) = \int_{\mathcal{J}_1} e^{-S/\hbar}\, dx + \int_{\mathcal{J}_{-1}} e^{-S/\hbar}\, dx$$

Analytically continues in $\hbar$.



$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ order 6

Relative Homology Local system

- $H_1(\mathbb{C}_x, \mathbb{B})$ forms a rk 2 local system on $\mathbb{C}^*_t$ with order 3 monodromy around zero.

- Bases change across "Stokes Rays", where there are morse flow lines between critical points.

- $[R]$ continues over triple cover of $\mathbb{C}^*_t$

Twisted de Rham complex

$X \, (= \mathbb{C}_x)$ affine variety, $dvol \in \Omega^{n,0}(X)$

$S \in \mathcal{O}(X)$

$$\Omega^*_{-S/\hbar}(X) := e^{-S/\hbar}\, \Omega^*_{alg}(X)$$

<u>Lemma</u> $\Omega^*_{-S/\hbar}(X) \simeq \Omega^*_{alg}(X)$, $d - \frac{dS}{\hbar}\wedge$

Special element $e^{-S/\hbar}\, dvol$

Perturbation Theory

$$S = \frac{1}{2}x^2 + \sum_{k=3}^{N} \frac{a_k}{k!} x^k$$

$$Z(\hbar) = \int e^{-S(x)/\hbar} dx = \hbar^{-1/2} \int e^{-S(\hbar^{1/2} x)/\hbar} = \hbar^{-1/2} \int e^{-\frac{1}{2}x^2 + \sum_{k=3}^{N} \frac{a_k}{k!} \hbar^{k/2-1} x^k}$$

$$= \hbar^{-1/2} \sqrt{\frac{\pi}{2}} \left(1 + c_1 \hbar^{1/2} + c_2 \hbar + \cdots \right)$$

Sum over feynman diagrams

Borel Transform

$$f(t) = \sum c_k t^k$$

$$\mathcal{B}f(s) := \sum \frac{c_k}{k!} s^k$$

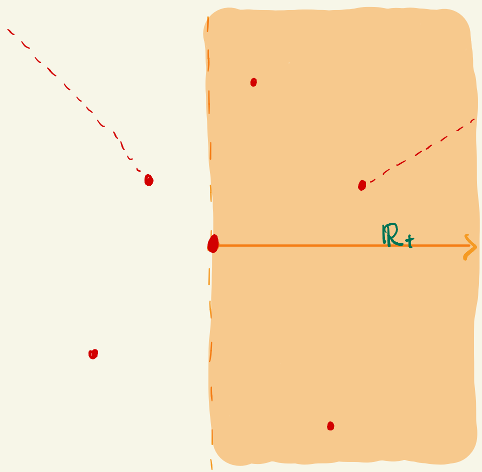
Calculus Excersize: If $f(t)$ entire,

$$f(t) = \frac{1}{t}\int_0^\infty e^{-s/t}\,\mathcal{B}f(s)\,ds$$

Borel Resummation

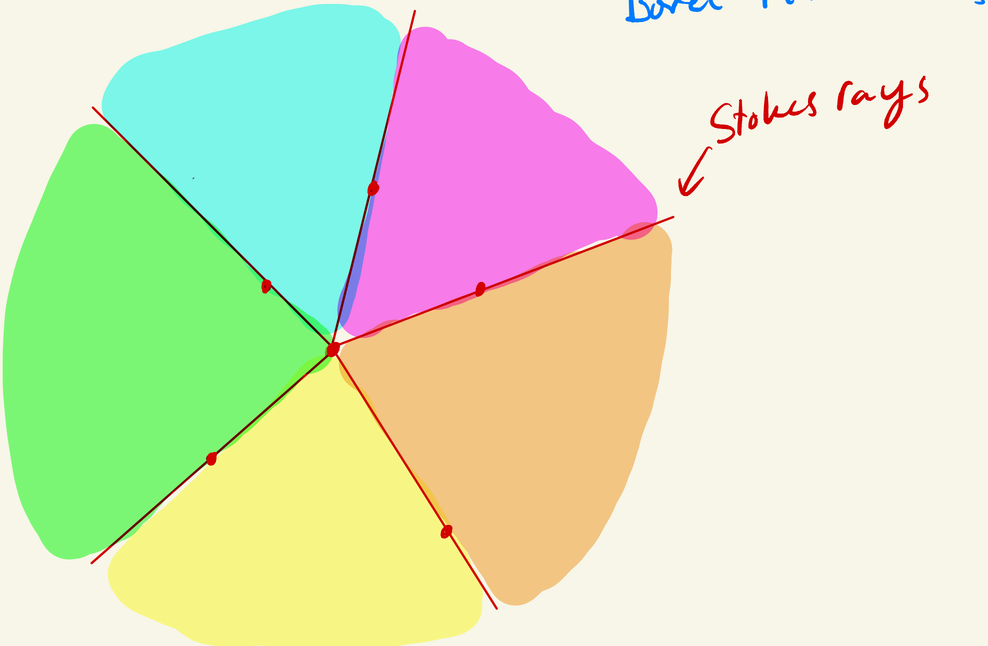
If $\mathcal{B}f(s)$ can be analytically continued along $\mathbb{R}_+$, define

$$f^{bs}(\hbar) := \frac{1}{\hbar}\int_{s=0}^{\infty} e^{-s/\hbar}\,\mathcal{B}f(s)\,ds \qquad (\mathrm{Re}(\hbar)>0)$$



Adjust Integration ray to phase of $\hbar$:

$$f^{\mathrm{rad}}(\hbar) := \frac{1}{\hbar}\int_{\mathrm{Arg}(\hbar)\mathbb{R}_+} e^{-S/\hbar}\,\mathcal{B}f(s)\,ds$$



Problem: Borel Resummation gives discontinuous answer!

Solution: must consider all critical points and combine.

Borel Transform has meaning:

$$Z(\hbar) = \int e^{-S(x)/\hbar}\,dx = \int e^{-s/\hbar}(S_* dx)$$

$$\begin{tikzcd} \mathbb{R}_x \arrow[r] \arrow[d, hook] & i\mathbb{R}_s \arrow[d, hook] \\ \mathbb{C}_x \arrow[r, "S"] & \mathbb{C}_s \end{tikzcd}$$

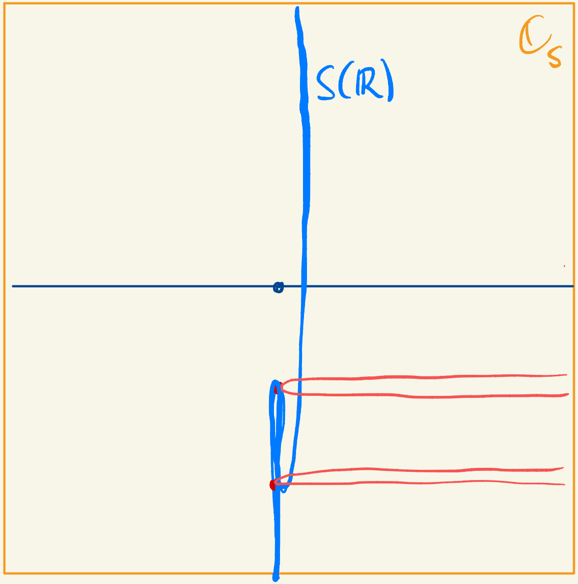
branched triple cover

$\Rightarrow$ $S_* dx$ is the Borel transform of $\hbar Z(\hbar)$

Consider "$S_* dx$" as a multivalued

1-form indexed by sheets of $S : \mathbb{C}_x \to \mathbb{C}_s$

$S(x) = i(x^3 - x)$



Picture of $[\mathbb{R}] = \mathcal{J}_1 + \mathcal{J}_{-1}$

in the Borel plane.

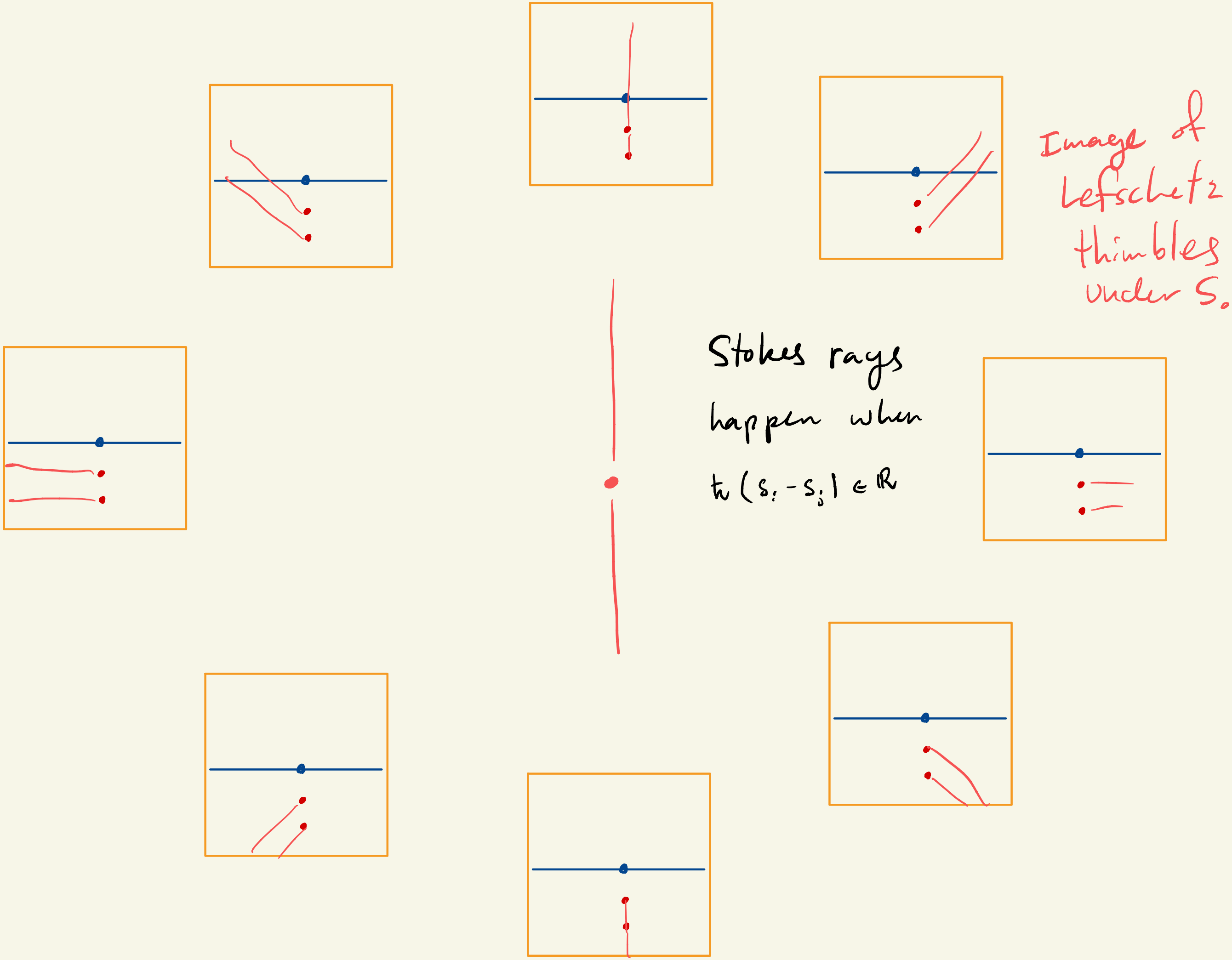


Image of Lefschetz thimbles under S_0
Stokes rays happen when $\mathrm{tr}(s_i - s_j) \in \mathbb{R}$

Recipe for analytic continuation:

① Write $[X_{\mathbb{R}}]$ in terms of Lefschetz thimbles

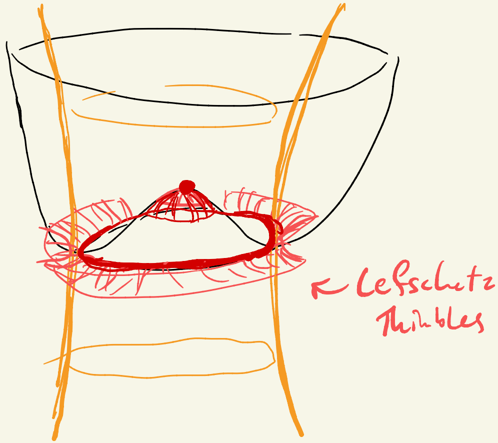
$$[X_{\mathbb{R}}] = \sum n_i \mathcal{J}_{\hbar}^{(i)}$$

$$Z(\hbar) = \sum n_i Z^{(i)}(\hbar)$$

② At stokes rays, apply stokes matrices to $\vec{n}$ to get loc. const. section of homology.

Stokes Matrices: $M_{ij} = \sum_{\substack{\text{flow lines} \\ \text{from } x_i \text{ to } x_j}} \pm 1$ $\quad x_i, x_j \in \mathrm{Crit}(S)$

$$S = (x^2+y^2)^2 - (x^2+y^2)$$



Must Choose $\frac{1}{2}$-dim cycle of $\mathrm{Crit}(S)$ to get $\dim_{\mathbb{R}} \mathcal{J}^{(i)} = d$

If $C^{(i)}$ orbit of symmetry of S there is a natural choice

Suppose

X variety

$S : X \to \mathbb{C}$ holomorphic

$G_{\mathbb{C}} \curvearrowright X$ symmetry of S

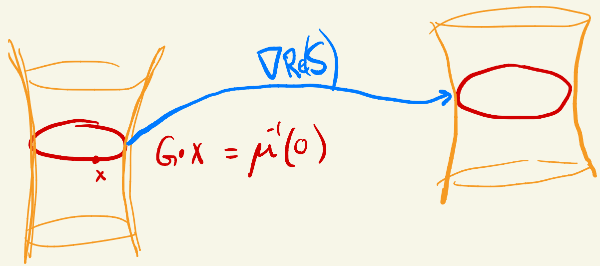
$\omega \in \Omega^{1,1}(X)$ Kähler form

$G \subset G_{\mathbb{C}}$ maximal compact, acting by hamiltonian isometries

$\mu : X \to \mathfrak{g}^*$ moment map

<u>Def</u> a $G_{\mathbb{C}}$ orbit $\mathcal{O}$ is semistable if it contains a point x with $\mu(x) = 0$.

$\Rightarrow \mathcal{O} = G_{\mathbb{C}} \cdot x \cong T^*(G \cdot x)$



<u>Lemma</u> μ preserved under $\nabla \mathrm{Re}(S)$

$\{\langle \mu, \xi \rangle, S\} = 0$

$\Rightarrow$ we only care about semistable critical loci.

Application to CS theory:

$$\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M^3 \end{array}$$

$P_{\mathbb{C}} :=$ associated $G_{\mathbb{C}}$ bundle

- $\mathrm{Conn}(P_{\mathbb{C}})$
- $\widetilde{CS} : \mathrm{Conn}(P_{\mathbb{C}}) \longrightarrow \mathbb{C}$
- $\mathrm{Aut}_0(P_{\mathbb{C}}) \curvearrowright \mathrm{Conn}(P_{\mathbb{C}})$

We can write $\Theta = A + i\phi$ where $A \in \mathrm{Conn}(P)$, $\phi \in \Omega^1(M^3, \mathfrak{g}_P)$)
So $\bar{\Theta} = A - i\phi$.

- $h = \int_{M^3} \langle \delta\bar{\Theta} \wedge * \delta\Theta \rangle$, $\omega = \int_{M^3} \langle \delta\phi \wedge * \delta A \rangle$
- $\mathrm{Aut}(P)$ acts preserving kähler str.
- $\mu(\Theta) = d_A * \phi$

(reduction to max compact)

Flat $G_{\mathbb{C}}$ connection semistable $\iff$ $\exists$ harmonic "metric"

Eg: trivial conn, hyp conn

Flow Equations:

$$d\, CS_\theta(\delta\theta) = \int_{M^3} \langle \delta\theta \wedge \Omega \rangle$$

here $\Omega = F_A - \phi \wedge \phi + i d_A \phi$ is the curvature of Θ

$$\nabla CS_\theta = \star \overline{\Omega}$$

$$= \star \left(F - \phi\wedge\phi - i d_A \phi \right)$$

Flow Equation: $\dot{\Theta} = - e^{-i\alpha} \star \overline{\Omega}$

$$\dot{A} = -\star\left(\cos\alpha\, (F - \phi\wedge\phi) - \sin\alpha\, d_A \phi \right)$$

$$\dot{\phi} = \star\left(\sin\alpha\, (F - \phi\wedge\phi) + \cos\alpha\, d_A\phi \right)$$

4D Interpretation

Consider $\Theta(t)$ as connection on $P_c \times \mathbb{R}$ over $M^3 \times \mathbb{R}$

$$\Omega^4 = dt \wedge \dot{\Theta} + \Omega^3$$

$$*\Omega^4 = \star\dot{\Theta} + dt \wedge \star\Omega^3$$

Self duality equations:

$$\dot{\Theta} = \star\Omega^3$$

$$\Leftrightarrow$$

$$\Omega = *\Omega$$

Kapustin-Witten equations:

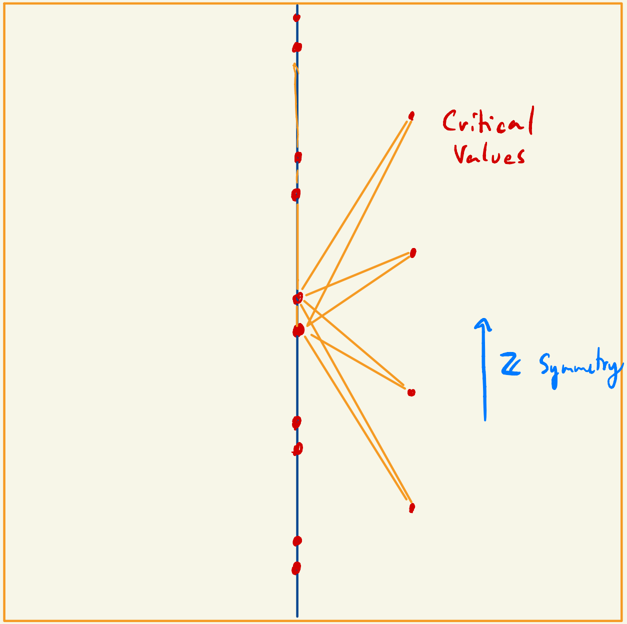
$$\dot{\Theta} = -e^{-i\alpha}\star\overline{\Omega^3}$$

$$\Leftrightarrow$$

$$\Omega = -e^{-i\alpha} * \overline{\Omega}$$

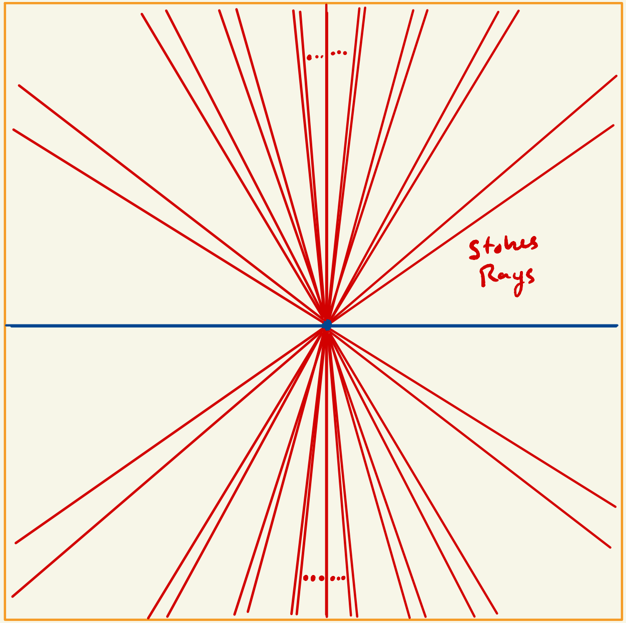
Should include $d_A \star \phi = 0$

Borel Plane



- Multivalued 1-form with expansions given by Borel transform of Perturbation theory.

$\hbar$ Plane



- $\mathcal{D}$-module with $\mathbb{Z}$-symm "twisted de-Rham cx"
- $\mathbb{Z}$-Local System, with jumping bases. Transition functions given by KW equations.
- Pairings from borel sumation of perterbation theory.